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Studies of eddy structure in the first few thousand feet of the atmosphere

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Part 2. A preliminary examination of the spectrum and scale of the vertical component at 2,000 feet.

by

R. A. Jones

(Meteorological Office)

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Studies of eddy structure in the first few thousand feet of the atmosphere

Part 2. A preliminary examination of the spectrum and scale of the vertical component at 2,000 feet

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SUMMARY

A preliminary calculation of the energy spectrum for the vertical component of eddy motion over the frequency range $(\frac{1}{75} - \frac{1}{5})$ cycles per sec. has been made, using observations of wind inclination at 2,000 feet over Cardington under near neutral conditions. The description of this calculation is preceded by a brief review of the principles of the various methods available for computing energy spectra, including the Fourier-transform method as developed and modified by Tukey (1950), the practical details of which are given in the appendices. Graphs and tables show the effect of the modifications on the resulting energy spectrum, and it is seen that the spectral curves decrease rapidly with increasing frequency over the greater part of the frequency range investigated. This variation with frequency agrees well with that deduced from measurements at similar heights from aircraft, and is broadly consistent with with the predictions of similarity theory.

An attempt was also made to investigate the vertical scale of turbulence around 2,000 feet through the Eulerian space-correlation coefficients, but the vertical space separations then attainable (up to 150 feet) were insufficient for a full specification of the correlation curve.

INTRODUCTION

In the study of atmospheric turbulence and diffusion it is desirable to estimate the turbulent energy of the various velocity components, and also to investigate the scale of turbulence as a means of assessing the magnitude of the eddies taking part in the turbulent motion. The total energy of the motion represents fluctuations in a wide range of frequencies (reciprocal of the period), and a more detailed knowledge of the eddy structure can be obtained if the fractions of the energy which are associated with the various frequencies are known. Published work to date on the evaluation of "energy spectra" (for example Panofsky and McCormick 1954, Webb 1955, and Panofsky and Van der Hoven 1956) and of the scale of turbulence (Howcroft 1956, MacCready 1953 and Gerhardt et alia 1952) is largely confined to the lower layers of the atmosphere. This paper describes a preliminary calculation of the energy spectrum and the scale of turbulence for the vertical velocity component, using some recent data on wind inclination at 2,000 feet. Discussion of the data and the calculation is preceded by a brief review of the principles of the various numerical methods available for computing energy spectra of velocity fluctuations, including the Fourier-transform method used here. The practical details of the method used are presented in appendices.

/I.

I. METHODS OF COMPUTING ENERGY SPECTRA

1. Moving averages

It is possible to get an idea of the energy distribution in turbulent motion by the crude method of forming a succession of moving averages of the original data with different periods (Panofsky and Singer 1951). The averaging process cuts out the shorter period fluctuations more or less effectively, leaving the longer period fluctuations. Hence, by choosing a succession of averaging periods, one can separate to some extent by subtraction the energy contributions possessed by eddies of a particular frequency range.

Given a sequence of observations X_i for i = 1, 2 ..... N, individual values of which are short-period means over δT sec., the simplest procedure is to form another sequence of running means Y_i of period T sec. (where T is some multiple of δT). Then $(X_i - Y_i)$ contains the contribution from the eddies with periods T to δT sec. or in the frequency range $1/\delta T$ to $1/T$ cycles per sec. This process can be repeated as often as desired, subject to limitation by the original period δT sec. and the number of observations N. It should be noted that the mean values of the parameter $(X_i - Y_i)$ in the separate sequences are not strictly zero, since in forming the mean of Y_i the end values of the original series have less weight than the remaining values. This is overcome by considering the energy to be contributed by fluctuations from the mean $\overline{(X_i - Y_i)}$ in each sequence. The energy contribution to the total energy of the motion from the respective frequency ranges is then determined by the mean square of the fluctuations from that mean.

The most important defect is that a moving average does not have a sharp energy cut-off, so that some of the energy assigned to a particular frequency range is actually contributed by fluctuations outside that range. This can be eliminated to some extent by forming weighted moving averages from the original series X_i, thereby sharpening the cut-off property in the frequency ranges. This modification is mentioned by Tukey (1951) in an unpublished paper, but has not been widely used by workers on energy spectra.

2. Fourier-transform method

Spectral analysis by this method is based on the theorem (Wiener 1930) that the cosine-transform of the autocorrelation function of a time-series is essentially equal to the spectrum of the original series. In the classical form (Priestley and Sheppard 1952)

$$F(n) = 4\int_0^\infty R(t).\cos 2\pi n t.\,dt; \qquad \int_0^\infty F(n)\,dn = 1 \qquad (1)$$

where R(t) is the autocorrelation coefficient after a time t, F(n)dn is the contribution to the energy present between the frequencies n and (n + dn) cycles per unit time. This transformation can be applied to a finite period T provided the period is taken sufficiently long for R(t) to become insignificant at all times greater than T. The accuracy of the results will increase with increasing length of the given series.

Assuming a continuous curve for the correlogram, energy spectra can be computed from equation (1) to varying degrees of approximation by the following three methods:-

(a) graphical integration of the product of the assumed correlation curve with a succession of cosine functions of known periods,

(b) the use of an appropriate integral approximation formula (e.g. Webb 1955),

(c) approximating the correlation function R(t) to a standard decay function (Webb 1955), evaluating the spectral estimates analytically, and subsequently applying a correction for the residue using a planimetric method (Crease and Tucker 1954).

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The extreme range of frequencies over which spectral estimates can be obtained is $[1/N\delta t - 1/2\delta t]$ cycles per sec. It has a lower bound at the low frequency end of the spectrum determined by the total length of the original record, and an upper bound at the high frequency end of the spectrum governed by the time interval δt between successive observations of the original data. Spectral analysis by Fourier-transform methods has a two-fold advantage over a straightforward Fourier analysis of the original data. Firstly, the computational time is shorter. Secondly, the correlation method gives the average power or energy over a range of frequency, in contrast to the energy present in each separate harmonic of Fourier analysis, with the result that it yields a 'smoothed' form of spectrum which is reproducible from a statistically similar set of data.

Tukey (1950) has presented a computational technique for evaluating the energy spectrum through equation (1) in a more realistic manner, which

(1) recognises the discontinuous nature of the correlogram,

(2) automatically limits the extreme frequency range which can be investigated,

(3) enables the statistical accuracy of the resultant spectral estimates to be determined,

(4) ultimately gives 'smoothed' spectral estimates representing average values over small finite frequency bands, and

(5) presents modifications to eliminate certain sources of error, dealt with in Appendix II under "aliasing", "prewhitening", and "averaging".

This method has been widely used by American workers (e.g. Panofsky and Van der Hoven 1954) and is used in the calculations in Part II of this paper. The computational procedure, together with the smoothing and correcting processes, are set out in some detail in the appendices.

Essentially the method consists firstly of forming values of the lag-products Q_k (lagged covariance) of the velocity fluctuations from the overall long-term mean values, thus

$$Q_k = \frac{1}{N-k}\left\{\sum_{i=1}^{N-k} X_i X_{i+k} - \frac{1}{N-k}\sum_{i=1}^{N-k} X_i \sum_{i=1}^{N-k} X_{i+k}\right\}; \quad k = 0, 1 \cdots m \qquad (2)$$

for successive values of the lag-time $k, \delta t$. Covariances are used instead of correlation coefficients as in equation (1), since Wiener's original transformation is in terms of covariances. Thus, complications due to the variance altering with the lag-number k (i.e. changing with time) do not arise.

The number of lags, m, should be chosen with care. An estimate of the best value of m can be obtained from 'sampling theory' (Tukey 1950) by the use of the formula

$$f = \text{number of degrees of freedom} = \frac{2(N - m/4)}{m} \qquad (3)$$

the reliability of the spectral estimates increasing with increasing number of degrees of freedom. However, if m is too large the computational work involved becomes prohibitive; also the estimates of spectral energy obtained are uncertain due to the low value of f resulting for a fixed number of observations N. If m is too small the resolution of spectral estimates at low frequencies is poor. The number of degrees of freedom, through m, imposes a further restriction upon the lowest frequency at which reasonably accurate spectral estimates are possible. Thus the extreme range of frequencies over which spectral estimates are obtainable becomes limited to $[1/2m\delta t - 1/2\delta t]$ cycles per sec. Values recommended by American workers in this field are $6 \leqslant m \leqslant 30$ and $f > 30$.

/Subjecting

Subjecting the lagged covariances Q_k to harmonic analysis finally yields values of the 'smoothed' spectral estimates U_h, for h = 0, 1..... m, which are related to the classical theory's energy densities through the equation

$$U_h \equiv F(n).\Delta n.Q_o \qquad (4)$$

where Q_o = total variance, $n = h/2m\delta t$ and $\Delta n \equiv 1/2m\delta t$. The 'smoothed' spectral estimate U_h represents an average total energy contribution over the frequency range $(h-1)/2m\delta t \leqslant n \leqslant (h+1)/2m\delta t$ for h = 1, 2 ..... (m - 1), while U_o and U_m are separately defined as the average energy contributions over the ranges $0 < n \leqslant 1/2m\delta t$ and $(m-1)/2m\delta t \leqslant n \leqslant 1/2\delta t$ respectively. The process of smoothing must inevitably lead to some distortion near h = 0 and m.

The values of U_h^{*}, corrected for averaging of the original data (see Appendix II), when plotted as a function of the frequency represent the energy spectrum curve, which has the property that the area beneath the curve, between any two frequencies, is a measure of the contribution to the total variance from that frequency range; and the total area bounded by the curve when extended to infinity represents the total variance. In order to examine very large ranges of frequency the spectral estimates can be conveniently plotted against abscissae of logarithm of frequency; if the spectral estimates are themselves multiplied by their representative frequencies, then the same relationship exists between the area under the curve and the variance contribution.

The final result U_h^{*} represents an estimate of the average energy over a finite frequency range obtained from a certain finite sample of data X_i (i = 1, ... N), and the result will vary from sample to sample under given external conditions. However, it is possible to determine the statistical reliability of U_h^{*}. Assuming that the original data X_i are normally distributed, the spectral estimates U_h^{*} from different samples (containing the same number of observations) will have a χ^2/f - distribution for f degrees of freedom given by equation (3) (Tukey 1950). Hence given the value of f, the range of variability of the estimate U_h^{*} in each frequency band, for desired confidence limits, can be determined by means of the usual Tables of the χ^2 - distribution found in any book on statistics. It should be noted that the number of degrees of freedom corresponding to the end estimates U_o, U_m are f/2 since the estimates are here only averages over half the previous frequency range.

3. Squared difference method

The method above is an exact one, involving a great deal of computation depending upon the number, m, of lagged products desired. Although the analysis can often be accomplished with desk calculating machines, for very large m the use of electronic computers becomes necessary. A simpler method suggested by Tukey (Cramer 1955) of obtaining rough spectral estimates of the energy contained within various frequency bands, is that of deriving a sequence of squared differences between successive observations. The method gives very limited resolution, especially at low frequencies.

Given the set of consecutive data in a time series X_i for i = 1, 2..... N, with a time-lag δt sec. between successive values, the procedure is to form the sequences of squared differences of consecutive values of the sum of 2^{p-1} non-overlapping terms for p = 1, 2, ..., thus

$$\left[d_i(1)\right]^2 = \left[X_{2i} - X_{2i-1}\right]^2$$

$$\left[d_i(2)\right]^2 = \left[\left\{X_2 2_i + X_2 2_{i-1}\right\} - \left\{X_2 2_{i-2} + X_2 2_{i-3}\right\}\right]^2 \qquad (5)$$

$$\left[d_i(p)\right]^2 = \left[\left\{X_2 p_i + \cdots + X_2 p_{i-(2^{p-1}-1)}\right\} - \left\{X_2 p_{i-2^{p-1}} + \cdots + X_2 p_{i-(2^p-1)}\right\}\right]^2$$

for i = 1, 2, ... $\frac{N}{2^p}$, N being an integral power of 2.

It can be shown (Cramer 1955) that mean spectral density

$$= \frac{1}{N} \sum_{i=1}^{N/2^p} \left[d_i(p) \right]^2, \quad \text{for } p = 1, 2 \text{ ----} \tag{6}$$

representing rough estimates over the frequency intervals $\left[\frac{1}{2^p \delta t} - \frac{1}{2^{p+1} \delta t} \right]$ cycles per sec., but the cut-off property between the various frequency bands is not sharply defined and thus results in limited resolution.

This method is subject to the same sources of error, "aliasing" and "averaging", as the Fourier-transform method, but to a greater degree; however it involves less computation. At low frequencies the minimum confidence limits, assuming a χ^2-distribution for maximum number of degrees of freedom (which equals the total number of i's), become very wide for a large p due to the small number of degrees of freedom. Thus for N = 512, one must have $p<9$ for reasonable accuracy of spectral estimates at the lower frequencies.

4. Composite spectra

It is possible to increase the extreme frequency range over which the energy spectrum is obtained (Griffith et alia 1956) without increasing the computational work prohibitively, by forming several sequences composed of various short-period non-overlapping mean values of the original data X_i. Thus originally the frequency range is $\left[\frac{1}{2m\delta t} - \frac{1}{2\delta t} \right]$ cycles per sec. for m lags, N observations and f degrees of freedom. Forming the non-overlapping means over $q\delta T$ sec., the spectral estimates for the range $\left[\frac{1}{2mq\delta T} - \frac{1}{2q\delta T} \right]$ cycles per sec. can be calculated. The value of m, upon which the increased frequency range depends, can be chosen subject to variation in f or the length of the original data X_i. This process can be repeated for different values of q. When combining the separate portions into a composite spectrum it is essential that the correction for the different averaging periods $q\delta T$ sec. in the various sequences is applied (see Appendix II). It is also important to note that the highest frequency estimates in each sequence are of doubtful accuracy due to "aliasing", but this can be overcome by choosing q so that the various frequency ranges overlap sufficiently to obviate the spurious values.

5. Estimation of the remaining energy from very high frequencies

The computed energy spectrum is limited at the high frequency end by the interval δt sec. between successive observations, the highest available frequency being $n_1 = 1/2\delta t$ cycles per sec. An estimate of the remaining energy contained in the frequency range from n_1 cycles per sec. to infinity can be obtained by extrapolation of the spectral curve to higher frequencies. It is usually found that a 'power-law' relation holds for that part of the spectral curve which decreases continuously with increasing frequency, resulting in a form

$$U(n) = A n^{-\beta} \tag{7}$$

where $A, \beta > 0$, are constants.

Then

$$\text{energy remainder } U(n) = \int_{n_1}^{\infty} A n^{-\beta} dn = \frac{A n_1^{-\beta+1}}{1-\beta} \text{ provided } \beta > 1 \tag{8}$$

Spectral curves of the form of equation (7) have been found with values of $\beta = 5/3$ (MacCready 1953) and $\beta = 2$ (Robinson 1953); and it should be noted that similarity theory for homogeneous turbulence predicts $\beta = 2$ for the inertial subrange of frequencies (Priestley and Sheppard 1952).

II. CALCULATIONS OF ENERGY SPECTRA AND SCALE OF TURBULENCE

The experimental data consist of consecutive observations of wind inclination at a height of about 2,000 feet over Cardington on 30th June, 1956. They are used in the present analysis to determine the Eulerian correlation coefficients, both with respect to time and to space (vertically), and partly in a trial computation of energy spectra using the Fourier-transform method developed and modified by Tukey. Although the results refer strictly to fluctuations of wind inclination, they are considered approximately applicable to the vertical component of eddy motion.

The observations were obtained from a vertical array of three wind vanes A, B, C attached to the cable of a tethered balloon, using a technique which is described in detail in a separate report (Jones and Butler, 1956). Defining the apparent wind inclinations at the heights of the instruments by ϕ_A, ϕ_B, ϕ_C respectively, the data were in the form of three synchronous continuous records of the variates ϕ_A, $\phi_B - \phi_A$, $\phi_C - \phi_B$ (+ve in this sense); and a continuous record of the vertical movement of the balloon was also obtained.

Three 10-minute periods were chosen for analysis and are listed below with the corresponding instrument spacing (the instrument A being in each case at a height of 2,000 feet).

Run	Period (B.S.T.)	Distances (feet)		
		AB	BC	CA
1	1017 - 1027	5	50	-
2	1128 - 1138	100	50	150
3	$1222\frac{1}{2}$ - $1232\frac{1}{2}$	20	50	70

Over the entire period of the three runs, the external meteorological conditions remained similar with 7/8 - 8/8 high stratocumulus with 4/8 shallow cumulus cloud beneath, a slight lapse rate (surface lapse between 4 metres and $\frac{1}{2}$ metre of 0.5 - 1.0°F), and a mean wind speed of approximately 8 metres per sec. at 2,000 feet.

Successive non-overlapping $2\frac{1}{2}$ sec. mean values of the variates ϕ_A, $\phi_B - \phi_A$, $\phi_C - \phi_B$, were extracted in each case, so providing sequences of 240 values. At the wind speeds obtaining the total response-time of instrument and recorder is governed by the response-time of the recorder and is approximately 0.2 sec., so that the use of $2\frac{1}{2}$ sec. mean values eliminates any errors due to instrumental time-lag. Approximate corrections for the spurious component arising due to the movement of the balloon were applied individually to the extracted values ϕ_A. It was not possible to apply a complete correction for balloon movement, because no data of fluctuations in the horizontal wind speed were available. However, it is believed that the error in not applying the full correction will be very small. In any case it is reasonable to assume that the corrections to $\phi_B - \phi_A$ and $\phi_C - \phi_B$ due to balloon movement are zero, since the distance involved between the instruments is at most 150 feet.

1. Energy spectrum

Run (2) was chosen for the computation of the energy spectrum of wind inclination at 2,000 feet because the autocorrelation coefficient decreased most rapidly in this case, becoming + 0.06 after a total time-lag of $37\frac{1}{2}$ sec. (i.e. m = 15) (Fig. I). 'Sampling theory' gives through equation (3) that f = number of degrees of freedom $\simeq 30$, so that m = 15 gives reasonably reliable estimates of the energy contributions from the respective frequency bands. The resolution is limited to a total frequency range of (1/75 - 1/5) cycles per sec.

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The procedure of Appendix I is now followed, forming the lagged covariances Q_k ($k = 0, 1 \ldots\ldots\ldots, 15$) and thence obtaining for $h = 0, 1, \ldots\ldots 15$ the 'apparent line powers' L_h, the 'smoothed' spectral estimates U_h, and the spectral estimates U_h^* corrected for 'averaging' (see Appendix II). The correction for averaging affects the spectral estimates at all frequencies, but is most marked at the highest frequencies. The individual values, related to a total variance or total energy of unity from all frequencies, are given in Table II. Spectral curves (Fig. II) are given with abscissae the logarithm of frequency and ordinates the spectral estimates (as a fraction of the total energy) multiplied by their respective frequencies in 1/75 cycles per sec., i.e. the ordinates are $nF(n)$. In this manner the area under the curve between any two frequencies represents directly the fraction of the variance, or energy, contributed by the frequencies in that interval. The 'smoothing' process eliminates the pronounced peak which occurs near the frequency of 1/15 cycles per sec.

The original data were next subjected to the "prewhitening" process of Appendix II with $b = 0.75$, and Fig.I shows the effect of this transformation on the correlation curve. The computational analysis was then repeated with the transformed data and the true spectral estimates L_h, U_h recovered by the application of equation (2) of Appendix II. These were again corrected for 'averaging' by equation (3) of Appendix II. The results are given in Table II and the spectral curve of 'corrected' U_h^*, plotted as a function of the logarithm of frequency, is shown in the same manner as before (Fig. II). As expected, the 'prewhitening' process does not appreciably alter the spectral estimates except at the lowest frequencies.

The spectral curves decrease rapidly with increasing frequency over the greater part of the frequency range investigated. The 'corrected' U_h^* values for the 'unprewhitened' case are found to fit a 'power law' of the frequency of the form

$$U^*(n) \propto n^{-\beta}$$

where $\beta = 1.59 \pm 0.20$ for the frequency range (1/75 - 15/75) cycles per sec., $\beta = 1.62 \pm 0.32$ for the frequency range (2/75 - 15/75) cycles per sec., and $\beta = 1.86 \pm 0.32$ for the frequency range (1/75 - 9/75) cycles per sec., representing 95% confidence limits in each case. These values show reasonable agreement with the inverse 'square law' predicted by the similarity theory of homogeneous turbulence for the 'inertial subrange' of frequencies (Priestley and Sheppard 1952).

It is interesting to note that data have recently been obtained (Crane and Chilton 1956) for the vertical component of eddy motion from an aeroplane flying at 1,700 feet in clear air with moderate turbulence (the measured apparent variance for the vertical component being 25 (feet per sec.)^2). Fig. III compares these results, extracted from graphs of two overlapping portions of the energy spectrum, with the Cardington results using the linear transformation $n = \bar{u}k$, where $\bar{u}$ is the mean wind speed at 2,000 feet (about 25 feet per sec.) and k is the wave-number (cycles per foot). The log-log plots of spectral density (cycles per foot)$^{-1}$ referred to a total variance of unity (i.e. the energy per unit interval of wave-number per unit variance) against wave-number show fairly good agreement.

2. Scale of turbulence

The investigation into the scale of turbulence was made through the spatial correlation function, characterised by the vertical separation between the instruments measuring the turbulent fluctuations.

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The spatial correlation coefficients R(z) were evaluated as cross-correlation coefficients between two sequences of wind inclination observations recorded simultaneously at two fixed points distance z apart, and for each of the separation distances z = 5, 20, 50, 70, 100, 150 feet, using the equalities listed below:-

$$\overline{\phi_A\phi_B} = \overline{\phi_A.(\phi_B-\phi_A)} + \overline{\phi_A^2}$$

$$\overline{\phi_B\phi_C} = \overline{(\phi_B-\phi_A).(\phi_C-\phi_B)} + \overline{\phi_A.(\phi_C-\phi_B)} + 2\overline{\phi_A.(\phi_B-\phi_A)} + \overline{(\phi_B-\phi_A)^2} + \overline{\phi_A^2}$$

$$\overline{\phi_C\phi_A} = \overline{(\phi_C-\phi_B).\phi_A} + \overline{(\phi_B-\phi_A).\phi_A} + \overline{\phi_A^2}$$

$$\overline{\phi_B^2} = \overline{(\phi_B-\phi_A)^2} + 2\overline{(\phi_B-\phi_A).\phi_A} + \overline{\phi_A^2}$$

$$\overline{\phi_C^2} = 2\overline{(\phi_C-\phi_B).(\phi_B-\phi_A)} + 2\overline{(\phi_C-\phi_B).\phi_A} + 2\overline{(\phi_B-\phi_A).\phi_A} + \overline{(\phi_C-\phi_B)^2} + \overline{(\phi_B-\phi_A)^2} + \overline{\phi_A^2}$$

and hence $R(z) = R(AB) = (\overline{\phi_A\phi_B}-\bar{\phi}_A\bar{\phi}_B)/(\overline{\phi_A^2}-(\bar{\phi}_A)^2)^{\frac{1}{2}}(\overline{\phi_B^2}-(\bar{\phi}_B)^2)^{\frac{1}{2}}$

the bars denoting temporal mean values.

The autocorrelation coefficients R(t) as a function of the lag-time t were also evaluated for the three runs separately (Fig. I), from the values of the inclination ϕ_A at 2,000 feet, and the individual values are given in Table I. Assuming the turbulence is isotropic and that the velocity fluctuations are small compared with the resultant mean velocity $\bar{u}$ at 2,000 feet, the transformation $z' = \bar{u}t$ can be used to convert the Eulerian time-correlation R(t) into the equivalent Eulerian space-correlation R(z'). The values of R(z) and ratios of R(z) to R(z') are shown below.

	Run 1		Run 2			Run 3		
z feet	5	50	50	100	150	20	50	70
R(z)	0.99	0.90	0.94	0.86	0.78	0.98	0.96	0.94
R(z)/R(z')	1.01	1.07	1.01	1.09	1.15	1.00	1.04	1.01

It is seen that the actual Eulerian space-correlation coefficient R(z) is always greater than the corresponding derived correlation coefficient R(z'). This could arise because the Lagrangian time-correlation coefficient over a distance of travel equal to z is less than unity, or because the turbulence is non-isotropic, or to a combination of both. However, the distances involved in the experiment ($z \leqslant 150$ feet) provide correlation coefficients falling only to 0.78 and correspond to time-lags of less than 6 sec. in wind speeds of about 8 metres per second, so that the ratios can be obtained for only a small portion of the auto-correlation curve. In order, therefore, to assess fully the scale of turbulence, distances much greater than 150 feet need to be considered.

Computation of energy spectra by the Fourier-transform method (after Tukey)

This represents a technique, recommended by Tukey (1950), for the computation of the energy spectrum of a finite series of observations. The original data are a series of consecutive observations of wind velocity X_i for i = 1, 2.....N, which may be instantaneous values or short-period mean values, with successive observations separated by a time interval δt sec. The procedure is to form values of the lag-products Q_k (covariances) of the fluctuations from the overall long-term mean value, thus

$$Q_k = \frac{1}{N-k}\left\{\sum_{i=1}^{N-k} X_i X_{i+k} - \frac{1}{N-k}\sum_{i=1}^{N-k} X_i \sum_{i=1}^{N-k} X_{i+k}\right\} \quad k = 0, 1 \cdots m \qquad (1)$$

for successive values of the lag-time $k\delta t$.

The chosen number of lags, m, is fairly critical. An estimate of m can be obtained from the 'sampling theory' (Tukey 1950) by the use of the formula

$$f = \text{number of degrees of freedom} = \frac{2(N - \frac{m}{4})}{m} \qquad (2)$$

the reliability of the spectral estimates increasing with increasing number of degrees of freedom. However, if m is too large the computational work becomes prohibitive; also the estimates of the spectral energy obtained are uncertain due to the low value of f resulting for a fixed number of observations N. If m is too small the resolution of spectral estimates at low frequencies is poor. The extreme range of frequencies over which reasonably accurate spectral estimates are obtainable is $(\frac{1}{2m\delta t} - \frac{1}{2\delta t})$ cycles per sec. Values recommended by American workers in this field are $6 \leqslant m \leqslant 30$ and $f > 30$.

The covariance function Q(t) is an even function of the time-lag t, and values Q_k are known at the (m+1) equally spaced points $0 \leqslant k \leqslant m$. For the finite range $0 \leqslant t \leqslant m\,\delta t$, harmonic analysis determines the values of Q(t) at the given (m+1) points, exactly in the form of the cosine series

$$Q(t) = \sum_{h=0}^{m} L_h \cos\frac{\pi h t}{m\delta t} \qquad (3)$$

where L_h are the contributions at equally spaced frequency bands centred at $n = 0, \frac{1}{2m\delta t}, \cdots\cdots \frac{1}{2\delta t}$, and

$$\left.\begin{aligned} L_0 &= \frac{1}{2m}(Q_0 + Q_m) + \frac{1}{m}\sum_{k=1}^{m-1} Q_k \\ L_h &= \frac{1}{m}Q_0 + \frac{2}{m}\sum_{k=1}^{m-1} Q_k \cos\frac{kh\pi}{m} + \frac{1}{m}Q_m \cos\pi h \;;\; h = 1, 2 \cdots (m-1) \\ L_m &= \frac{1}{2m}(Q_0 + (-1)^m Q_m) + \frac{1}{m}\sum_{k=1}^{m-1}(-1)^k Q_k \end{aligned}\right\} \qquad (4)$$

and it is implicit that Q_0 = total variance = $\sum_{h=0}^{m} L_h$. The quantities L_h

/are

are termed "apparent line powers" and are discussed below.

In this way no assumption is made about the shape of the covariance curve, only those values of Q_k actually obtained by computation being used. The analysis leads to more accurate spectral estimates the greater the range of n, i.e. the greater the value of m.

If P(n) is the energy density at a frequency n, i.e. the energy contribution per unit frequency, and Q_k is the covariance, then the inverse Fourier-cosine transformation (Wiener 1930) gives

$$Q_k = \int_0^\infty \cos 2\pi n k \delta t \,.\, P(n) \,.\, dn \qquad (5)$$

$$\equiv \sum_n \cos 2\pi n k \delta t \,.\, P(n) \,.\, \Delta n$$

Substituting the values for Q_k from (5), into equations (4) for the 'line powers' L_h, give

$$L_h = \int_0^\infty l_h(n) \,.\, P(n)\, dn \qquad \text{for } h = 0, 1 \cdots m \qquad (6)$$

$$\equiv \sum_n l_h(n) \,.\, P(n) \,.\, \Delta n$$

where $l_h(n) = \frac{1}{m} + \frac{2}{m} \sum_{k=1}^{m-1} \cos \frac{kh\pi}{m} \,.\, \cos 2\pi k n \,.\, \delta t + \frac{1}{m} \cos h\pi \cos 2\pi m n \delta t$

Thus the 'apparent line power' L_h represents the energy weighted according to the function $l_h(n)$, which has the values of near unity at $n = \frac{h}{2m\delta t}$ and near zero elsewhere over the frequency range n; and so L_h can be considered an average estimate over the range $(h-\frac{1}{2})/2m\delta t \leq n \leq (h+\frac{1}{2})/2m\delta t$; except for the 'end values' L_0 and L_m which are averages over the frequency ranges $0 \leq n \leq \frac{1}{4m\delta t}$ and $(m-\frac{1}{2})/2m\delta t \leq n \leq \frac{1}{2\delta t}$ respectively.

In terms of the classical theory, the 'line powers' are effectively equivalent to

$$L_h \equiv F(n) \,.\, \Delta n \,.\, Q_0 \qquad \text{where } n = \frac{h}{2m\delta t} \qquad (7)$$

The curve of 'apparent line power' as a function of frequency oscillates considerably from point to point. An averaging process is, therefore, employed in which irregularities of the spectral curve, which occur due to the use of a finite instead of an infinite set of data, are 'smoothed' but the smoothing occurs at the expense of resolution in all the frequency bands. Experience has shown that the best results are given by the following 3-term weighted average:

$$\left.\begin{aligned} U_0 &= 0.54\, L_0 + 0.46\, L_1 \\ U_h &= 0.54\, L_h + 0.23(L_{h-1} + L_{h+1}) \quad \text{for } h = 1, 2 \cdots (m-1) \\ U_m &= 0.54\, L_m + 0.46\, L_{m-1} \end{aligned}\right\} \qquad (8)$$

U_h represents the average total energy contribution over the frequency range $(h-1)/2m\delta t \leqslant n \leqslant (h+1)/2m\delta t$; except the end values U_o and U_m which are averages over the frequency ranges

$$0 < n \leqslant \frac{1}{2m\delta t} \quad \text{and} \quad (m-1)\frac{1}{2m\delta t} \leqslant n < \frac{1}{2\delta t} \quad \text{respectively.}$$

Errors and modifications of the Fourier-transform method

(a) Aliasing

The effect known as 'aliasing' has been mentioned rather vaguely in a number of papers, and refers to the fact that there is present at the relatively lower frequencies 'apparent energy' which is really due to oscillations of frequency greater than $\frac{1}{2\,\delta t}$ cycles per sec. The time-interval δt sec. between consecutive observations determines the resolution of the spectrum and the highest frequency at which reasonably accurate estimates can be obtained. Although the error due to 'aliasing' cannot be wholly eliminated it can be minimised by making the time-interval δt as small as possible, its limiting value being the short-term averaging period δT of the original observations. The averaging period δT sec. also has a lower bound, since $2\delta T$ must not be less than the natural period (or not less than approximately five times the response time) of the instrument measuring the observations, in order to eliminate high frequency oscillations which are not faithfully recorded by the instrument.

For a given spectrum the error due to 'aliasing' increases with increasing frequency, and so it is current practice to discard the spectral estimates at the highest frequencies. This source of error is not so important in spectra of horizontal wind components, since most of the energy in this case is contained in the low frequency or long period eddies.

(b) Prewhitening

When the spectrum varies rapidly as a function of the frequency, spectral analysis by the Fourier-transform method yields poor estimates of the spectral energy contributions. This is particularly serious in the case of spectra of horizontal velocity components, the computations often resulting in negative spectral estimates at high frequencies. It is rectified by a process called "prewhitening" (Panofsky 1954) which entails forming a new series with a spectrum in which high frequency fluctuations are emphasised relative to the low frequency fluctuations. In practice a new series of consecutive observations is derived from the original one by means of the linear transformation

$$Y_i = X_i - bX_{i-1} \qquad \text{where } 0 < b \leq 1 \qquad (1)$$

$$\text{for } i = 1, 2 \ldots\ldots N$$

The constant $b = 0.75$ is usually chosen, and it is assumed that $\bar{X}_i = \bar{X}_{i-1}$ where bars denote overall long-term mean values. The whole computational procedure is now carried out with the new series Y_i, forming lagged covariances and ultimately obtaining 'prewhitened' spectral estimates V_h. The true spectral estimates, U_h, are then recovered, using the transformation

$$U_h = \frac{V_h}{1 + b^2 - 2b\cos\frac{\pi h}{m}} \qquad (2)$$

$$= \frac{V_h}{1{\cdot}5625 - 1{\cdot}5\cos\frac{\pi h}{m}} \qquad \text{for } b = 0.75.$$

(c) Averaging

An error is introduced into the spectral estimates when the original velocity data extracted from continuous records are mean values over a certain finite period δT sec. instead of instantaneous values. This causes the

effective amplitudes of the high frequency oscillations to be partially suppressed to an extent which depends upon the relation between the averaging period δT sec. and the actual period of oscillation under consideration. For example, the apparent velocity amplitude is damped to 0.64 of its true value when the averaging period δT is half the period of oscillation. In general the velocity field can be represented by the expression

$$V(t) = \int_{-\infty}^{\infty} S(n).e^{2\pi i n t}\, dn \,,$$ where n is the frequency of the harmonic oscillations into which the record is resolved and s(n) is the amplitude at frequency n. The effect of using average values over a finite time δT sec. is, therefore, to suppress the amplitude s(n) by a factor

$$\frac{1}{\delta T}\int_{-\frac{\delta T}{2}}^{\frac{\delta T}{2}} e^{2\pi i n t}\, dt = \frac{\sin n\pi\delta T}{n\pi\delta T} \,.$$

Hence the resolved spectral estimates, which depend upon the square of the velocity amplitudes, are corrected for 'averaging' according to the equation

$$U^{2}(n)_{\text{corrected}} = U(n)_{\text{average}} \times \left(\frac{n\pi\delta T}{\sin n\pi\delta T}\right)^{2} \qquad (3)$$

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TABLE I

Eulerian-time correlation coefficients (autocorrelations) R(t) for wind inclination at 2,000 feet

Run	Period (B.S.T.)	Lag number k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1017	1027	1.0	.79	.58	.46	.40	.33	.30	.32	.33	.32	.28						
2	1128	1138	1.0	.88	.72	.58	.47	.41	.32	.25	.18	.14	.13	.13	.13	.11	.09	.06	.03
3	1222½	1232½	1.0	.93	.87	.81	.77	.70	.62	.55	.51	.47	.42						
Arithmetic mean			1.0	.87	.72	.62	.55	.48	.41	.37	.34	.31	.28						

(Time-interval for 1 lag = 2½ sec.)

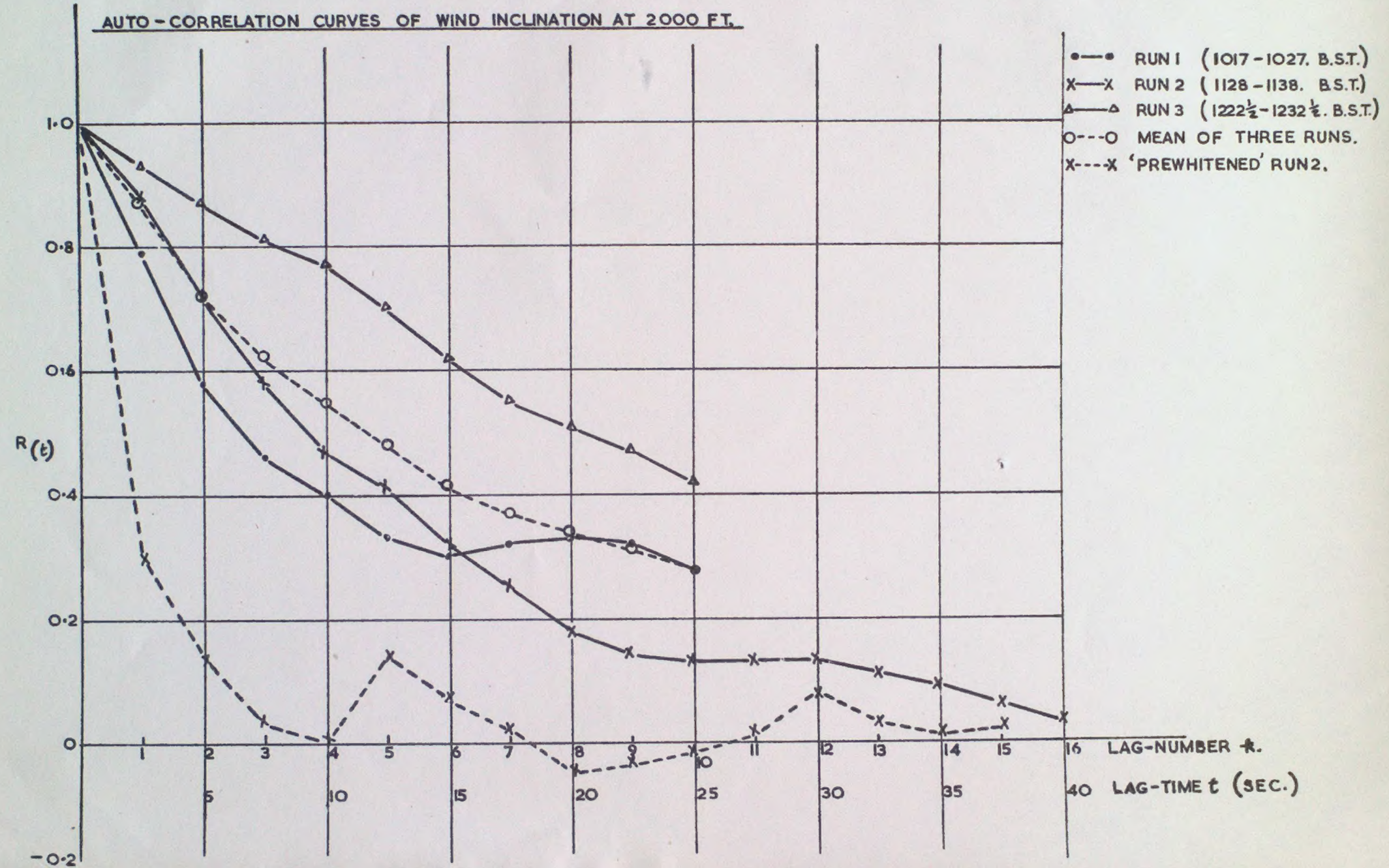
TABLE II

Spectral analysis for run 2 (see App. I for meaning of symbols)

h	Frequency n cycles/hr.	Average spectral estimates (fraction of total variance)			$hU_h^* \equiv nF(n)$	Prewhitened process			$hU_h^* \equiv nF(n)$
						Average spectral estimates			
		L_h	U_h	corrected U_h^*		L_h	U_h	U_h^*	
0	0	.3331	.3369	.3240	0	.3403	.4006	.3873	0
1	48	.3432	.2942	.2855	.2855	.3341	.2915	.2844	.2844
2	96	.1435	.1688	.1639	.3278	.1564	.1374	.1343	.2686
3	144	.0559	.0693	.0689	.2067	.0501	.0559	.0558	.1674
4	192	.0277	.0365	.0372	.1488	.0313	.0341	.0350	.1400
5	240	.0381	.0304	.0321	.1605	.0345	.0273	.0290	.1450
6	288	.0154	.0193	.0211	.1266	.0158	.0160	.0177	.1062
7	336	.0097	.0097	.0112	.0784	.0080	.0080	.0093	.0651
8	384	.0039	.0053	.0065	.0520	.0043	.0044	.0054	.0432
9	432	.0042	.0045	.0059	.0531	.0028	.0035	.0045	.0405
10	480	.0060	.0052	.0073	.0730	.0049	.0040	.0057	.0570
11	528	.0045	.0050	.0076	.0836	.0039	.0043	.0065	.0715
12	576	.0053	.0050	.0084	.1008	.0052	.0045	.0076	.0912
13	624	.0049	.0046	.0085	.1105	.0043	.0040	.0074	.0962
14	672	.0031	.0031	.0066	.0924	.0028	.0027	.0056	.0784
15	720	.0014	.0022	.0052	.0780	.0014	.0019	.0045	.0675

Total variance = Q_o = 29.1 (degrees)^2

FIG: 1.



M.R.P. 1044

ENERGY SPECTRA COMPUTED FOR RUN 2.

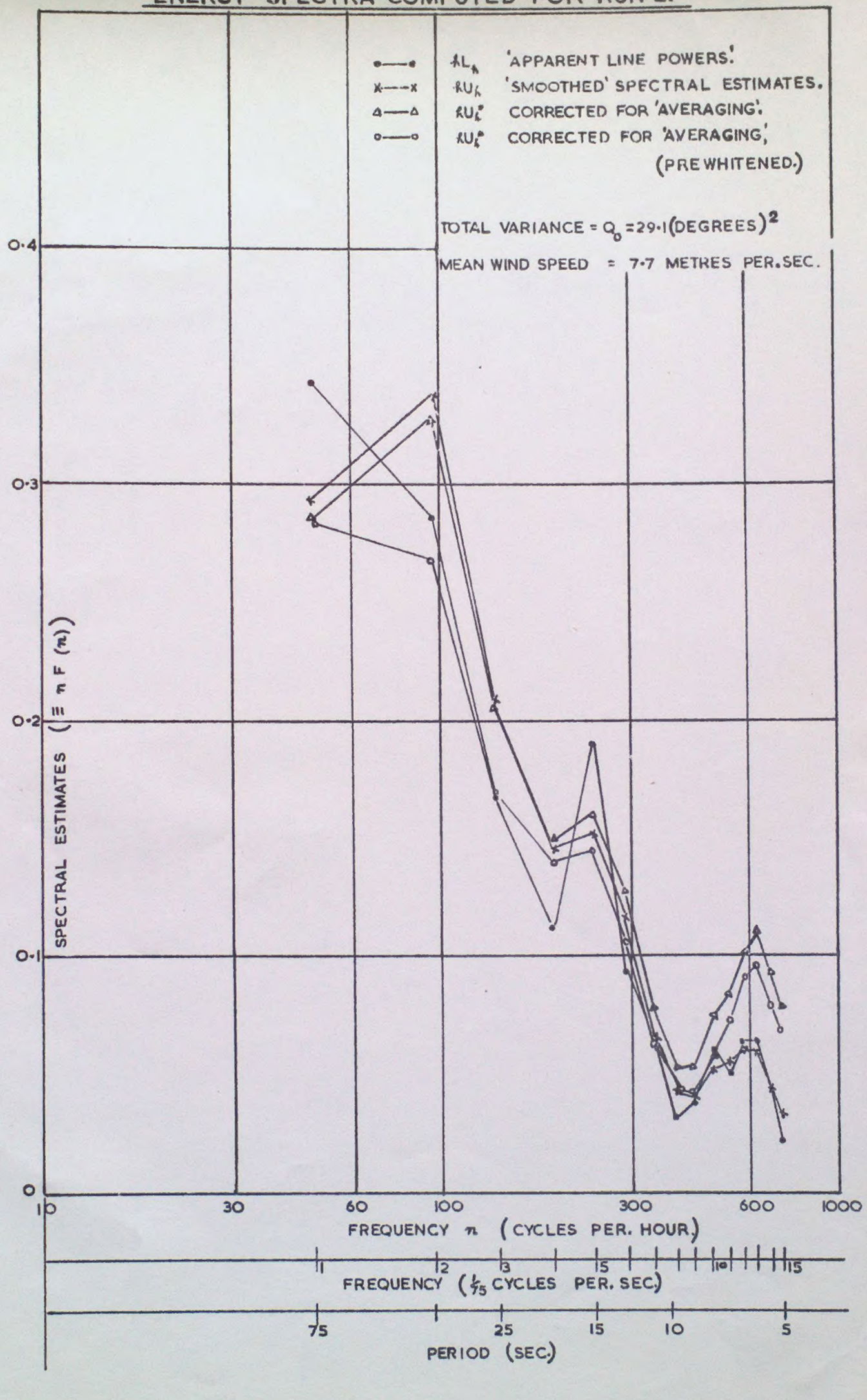


FIG II.

COMPARISON OF RUN 2 WITH DATA DUE TO CRANE AND CHILTON

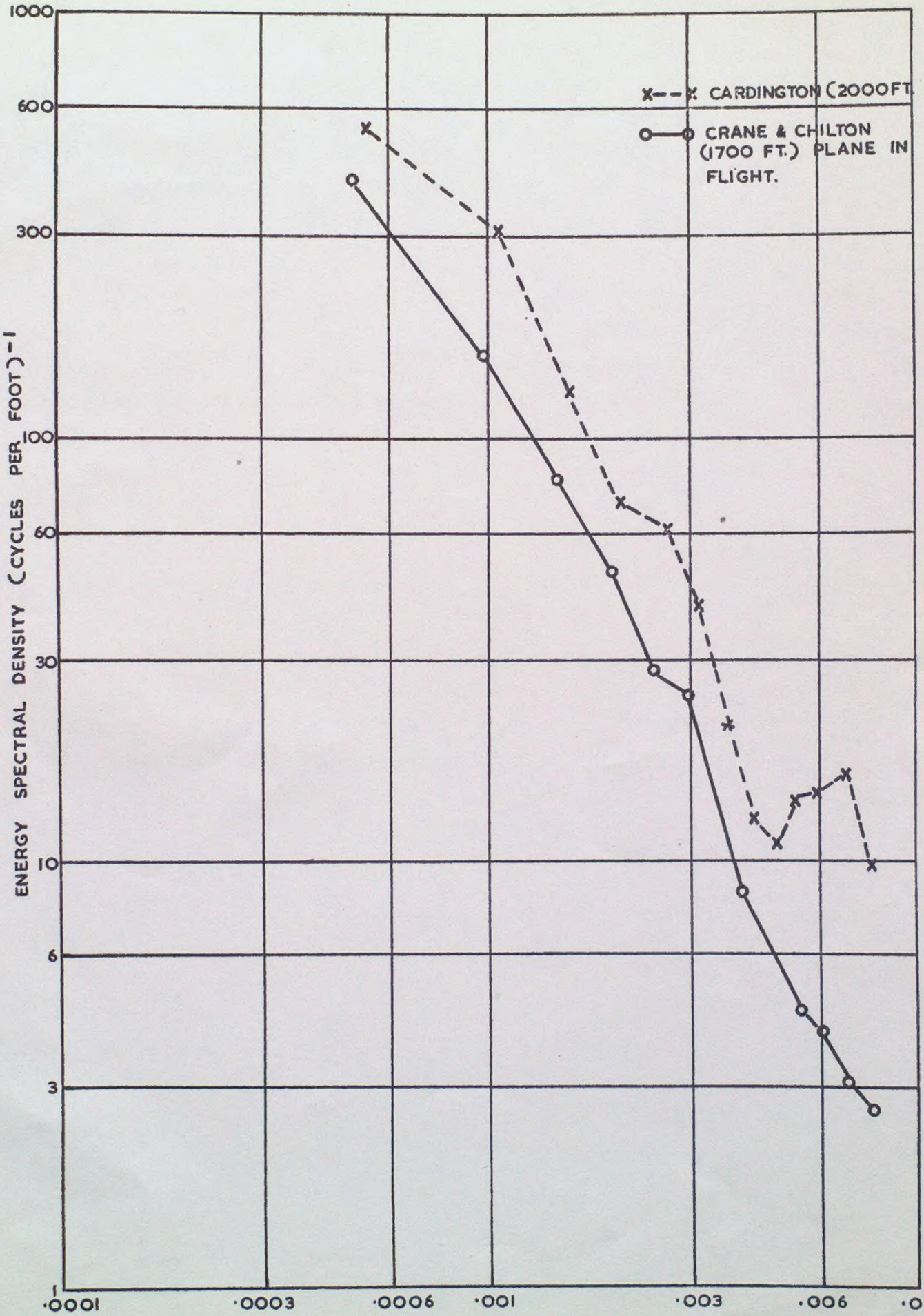


FIG. III